CHARACTERISTIC VALUE (EIGEN-VALUE) PROBLEMS

In many engineering fields and applications, including vibration, heat transfer, elasticity, and electromagnetics, we face a special class of problems known as boundary-value problems, or characteristics value problems. Typical characteristic of this type of problem is that it is described or formulated in terms of a second order differential equation and the desirable or achievable solution is required to satisfy two boundary conditions.

For example, let us consider the following second order differential equation

$$\frac{d^2y}{dx^2}+k^2y=0$$

and we which to obtain a solution for $y(x)$ subject to the boundary conditions

$$y(0) = 0,\ y(1) = 0$$

As you will note as we carry out this solution that the final solution $y(x)$ will only be possible for specific values of the parameter k and that these values of k are detected by the required boundary conditions. This is why these types of problems are known as "characteristic value" or "Eigenvalue" problems. Let us illustrate the solution procedure by solving the above equation analytically first, and we will then move to the finite difference implementation. A possible form of the analytical solution may take the form

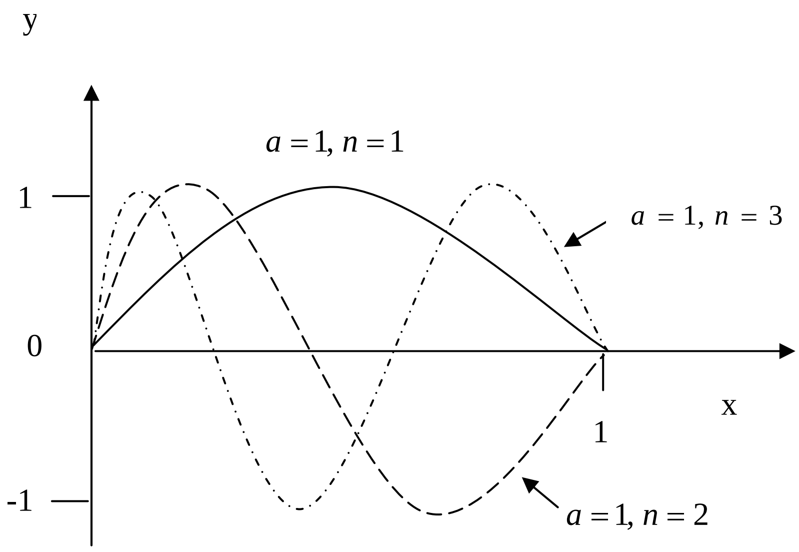
$$y = A \sin kx + B \cos kx$$

where A and B are constants to be determined. We can easily verify this form of solution by substituting into the differential equation. Substituting $y(0) = 0$, we obtain

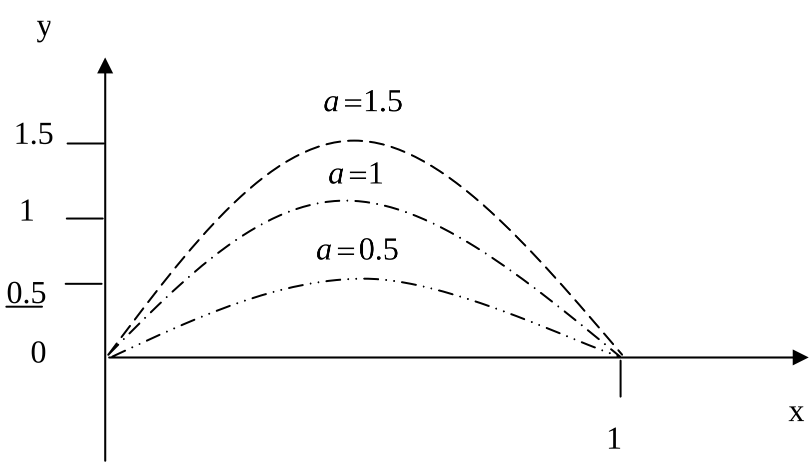
$$0 = A \sin 0 + B \cos 0$$

and to satisfy this equation $B = 0$. So now we have $y(x) = A \sin kx$.

We also need to satisfy $y(1) = A \sin k = 0$. We cannot select $A = 0$, as we will have no solution, so the only remaining option is that $k = \pm n\pi$ as $\sin(\pm n\pi) = 0$.
The point is this, the solution is only possible for discrete (multiplicative) values of k, hence the characteristic – value problem. The figure below illustrates the solution for various values of k.



Possible $y(x)$ for $a = 1$
and various values of n



Solution for various
values of a and
all are for $n = 1$

APPLICATION TO THE FINITE DIFFERENCE METHOD

To solve a boundary value problem using FD method, we as usual replace the differential equation by a finite difference equation using the central difference method. We subdivide the region of interest into a suitable number of equal (or unequal) subintervals, and write the difference equation at each point where the value of the function is unknown. The only difference between the procedure when we are using boundary value problems that the resulting matrix equation will take the form

$$[A - \lambda I][\Phi] = 0$$

where I is a unity diagonal matrix, A is the difference equation matrix, and $[\Phi]$ is the unknown potentials vector matrix. Therefore, in this case, in addition to the need to determine the Φ distribution in the domain of interest, we need to identify specific values of λ (eigenvalues) and for each value of λ, we determine the associated Φ distribution (eigenfunction). Let us illustrate this with examples.

Eigenvalues of a Matrix:

In waveguide type of problems, we are required to solve characteristic equations in which both the eigenvalues and eigenfunctions need to be determined. These equations may take the form

$$Ax = \lambda x$$

where A is a square matrix, x is the characteristic vector (Eigen function), and λ s are the eigenvalues. In this section, we will demonstrate two methods for determining these solutions.

A. Determinant Method

$Ax = \lambda x$ is equivalent to $(A - \lambda I)x = o$ and for a solution to exist, it is necessary that $|A - \lambda I|$ determinant be zero. This results in a polynomial in λ, and the roots will be the eigenvalues.

Example:

$$\begin{bmatrix} 4 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \lambda \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \rightarrow \begin{bmatrix} 4-\lambda & 1 & 0 \\ 0 & 2-\lambda & 1 \\ 0 & 0 & -1-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

The determinant of A, which is also known as the characteristic equation is

$$(4-\lambda)(2-\lambda)(1+\lambda) = 0 \rightarrow \lambda^3 - 5\lambda^2 + 2\lambda + 8 = 0$$

and the roots (eigenvalues) are

$$\lambda_1 = 4, \quad \lambda_2 = 2, \quad \lambda_3 = -1$$

There will be an eigenvector $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ for each of these eigenvalues:

$$\begin{bmatrix} 4 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 4 \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \Longrightarrow \begin{matrix} 4x_1 & + & x_2 & = & 4x_1 \\ 2x_2 & + & x_3 & = & 4x_2 \\ & & -x_3 & = & 4x_3 \end{matrix} \rightarrow \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

Also

$$\begin{bmatrix} 4 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 2 \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \Longrightarrow \begin{matrix} 4x_1 & + & x_2 & = & 2x_1 \\ 2x_2 & + & x_3 & = & 2x_2 \\ & & -x_3 & = & 2x_3 \end{matrix} \rightarrow \begin{bmatrix} -\frac{1}{2} \\ 1 \\ 0 \end{bmatrix}$$

and

$$\begin{bmatrix} 4 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = -1 \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \Longrightarrow \begin{matrix} 4x_1 & + & x_2 & = & x_1 \\ 2x_2 & + & x_3 & = & x_2 \\ & & -x_3 & = & -x_3 \end{matrix} \rightarrow \begin{bmatrix} -\frac{1}{15} \\ -\frac{1}{3} \\ 1 \end{bmatrix}$$

The eigenvectors are then $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -\frac{1}{2} \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -\frac{1}{15} \\ -\frac{1}{3} \\ 1 \end{bmatrix}$

B. <u>The Power Method</u>

This method is based on repeated multiplication of an initially assumed eigenvector by both sides of the matrix equation. After each step, the resulting new eigenvector is normalized by making its largest component equal to unity. After sufficient number of multiplication, the eigenvalue of largest magnitude would emerge as you can see from the following examples.

<u>Example</u>:

$$\begin{bmatrix} 4 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \lambda \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Start by assuming $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

Then

$$\begin{bmatrix} 4 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 5 \\ 3 \\ -1 \end{bmatrix} \xrightarrow[5]{\text{Divide by}} 5\begin{bmatrix} 1 \\ \frac{3}{5} \\ -\frac{1}{5} \end{bmatrix}$$

$$\begin{bmatrix} 4 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ \frac{3}{5} \\ -\frac{1}{5} \end{bmatrix} = \begin{bmatrix} 4.6 \\ 1 \\ \frac{1}{5} \end{bmatrix} \longrightarrow 4.6\begin{bmatrix} 1 \\ 0.217 \\ 0.0435 \end{bmatrix}$$

$$\begin{bmatrix} 4 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 0.217 \\ 0.0435 \end{bmatrix} \longrightarrow 4.217\begin{bmatrix} 1 \\ 0.1134 \\ -0.0103 \end{bmatrix}$$

$$\begin{bmatrix} 4 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 0.1134 \\ -0.0103 \end{bmatrix} \longrightarrow 4.1134\begin{bmatrix} 1 \\ 0.0526 \\ 0.0025 \end{bmatrix}$$

It may be seen that as we continue with this process an eigenvalue of 4 will emerge and a corresponding eigenvector $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ will be associated with it.