**FINITE DIFFERENCE METHOD:**
**SUPPLEMENTAL HANDOUT**

## Unequal Arms and Finite Difference Representation of $\nabla^2$ Operator

When the boundary of the region of interest is not such that the mesh can be drawn to have the boundary coincide with the nodes of the mesh, we must proceed differently at points near the boundary. Specifically, we need to consider using unequal arms in the finite difference mesh. Consider the general case of a group of five points whose spacing is nonuniform, arranged in an unequal-armed star. We represent each distance by $\alpha_i h$, where $\alpha_i$ is the fraction of the standard spacing $h$ that the particular distance represents as shown in Fig. 1.

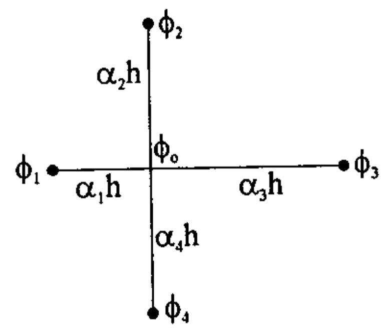


Fig. 1. Unequal-armed five-point star.

At the midpoint between $\phi_3$ and $\phi_o$, the first derivative is given by:

$$\left.\frac{\partial \phi}{\partial x}\right|_{0-3} = \frac{\phi_3 - \phi_o}{\alpha_3 h}$$

$$\left.\frac{\partial \phi}{\partial x}\right|_{1-0} = \frac{\phi_o - \phi_1}{\alpha_1 h}$$

The second-order derivative at the origin is given by

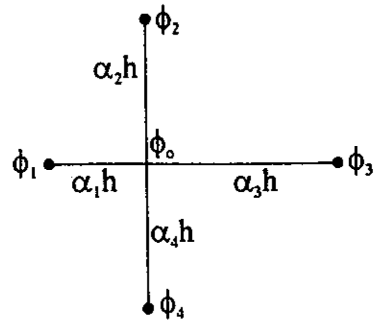


Fig. 1. Unequal-armed five-point star.

$$\frac{\partial^2\phi}{\partial x^2}=\frac{\left(\dfrac{\phi_3-\phi_o}{\alpha_3 h}\right)-\left(\dfrac{\phi_o-\phi_1}{\alpha_1 h}\right)}{\frac{1}{2}(\alpha_1+\alpha_3)h}$$

$$=\frac{2}{h^2}\left[\frac{\phi_1-\phi_o}{\alpha_1(\alpha_1+\alpha_3)}+\frac{\phi_3-\phi_o}{\alpha_3(\alpha_1+\alpha_3)}\right]$$

Similarly,

$$\frac{\partial^2\phi}{\partial y^2}=\frac{2}{h^2}\left[\frac{\phi_2-\phi_o}{\alpha_2(\alpha_2+\alpha_4)}+\frac{\phi_4-\phi_o}{\alpha_4(\alpha_2+\alpha_4)}\right]$$

The Laplacian operation is hence given by:

$$\nabla^2\phi=\frac{2}{h^2}\left[\frac{\phi_1}{\alpha_1(\alpha_1+\alpha_3)}+\frac{\phi_2}{\alpha_2(\alpha_2+\alpha_4)}+\frac{\phi_3}{\alpha_3(\alpha_1+\alpha_3)}+\frac{\phi_4}{\alpha_4(\alpha_2+\alpha_4)}-\left(\frac{1}{\alpha_1\alpha_3}+\frac{1}{\alpha_2\alpha_4}\right)\phi_o\right] \quad (1)$$

We use the operator of Eq. 1 for points adjacent to boundary points when the boundary points do not coincide with the mesh, instead of our standard operator. An example of this situation is when the geometry involves cylindrical shapes.