EXAMPLE 4.14

Consider the rectangular region shown in Figure 4.18a. The electric potential is specified on the conducting boundaries. Use the finite difference representation to solve for the potential distribution within this region.

Solution

The electric potential everywhere in the rectangular region should satisfy Laplace's equation. In this case of simple geometry, the analytical solution (e.g., using separation of variables) of Laplace's equation is possible. We will, however, use this simple example to develop a numerical solution procedure that may be used to solve much more complicated geometries. Using a numerical solution means we will define Φ in the rectangular region of interest by calculating its values at discrete points, the nodes of a mesh. The step-by-step solution procedure includes the following:

1. Layout a coarse square mesh and identify the nodes at which the electric potential is to be calculated. The geometry of a 2×4 mesh is shown in Figure 4.18b. The value of h (mesh size) in this case is $h = 5$ cm.
2. Replace Laplace's equation by its finite difference representation.

$$\frac{1}{h^2}(\Phi_{i+1,j} + \Phi_{i-1,j} + \Phi_{i,j+1} + \Phi_{i,j-1} - 4\Phi_{i,j}) = 0$$

   $\Phi_{i,j}$ are the discrete values of the potential at points (nodes) within the domain of interest.

3. Apply the difference equation in step 2 at each node. At node 1,

$$\frac{1}{(0.05)^2}(0 + 0 + 0 + \Phi_2 - 4\Phi_1) = 0$$

or

$$4\Phi_1 = \Phi_2 \tag{4.50}$$

At node 2,

$$\frac{1}{(0.05)^2}(\Phi_1 + \Phi_3 + 0 + 0 - 4\Phi_2) = 0$$

or

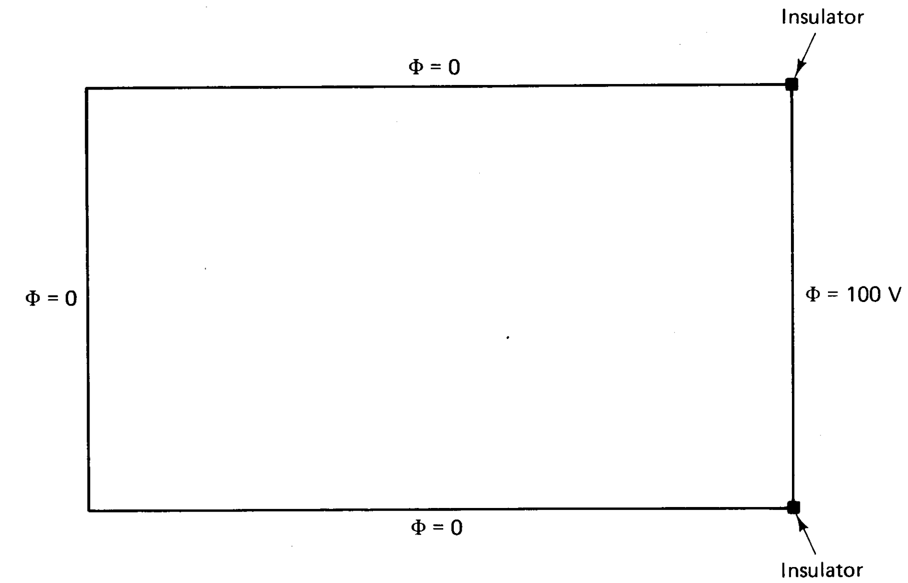
$$\Phi_1 - 4\Phi_2 + \Phi_3 = 0 \tag{4.51}$$

At node 3,

$$\frac{1}{(0.05)^2}(\Phi_2 + 100 - 4\Phi_3) = 0$$

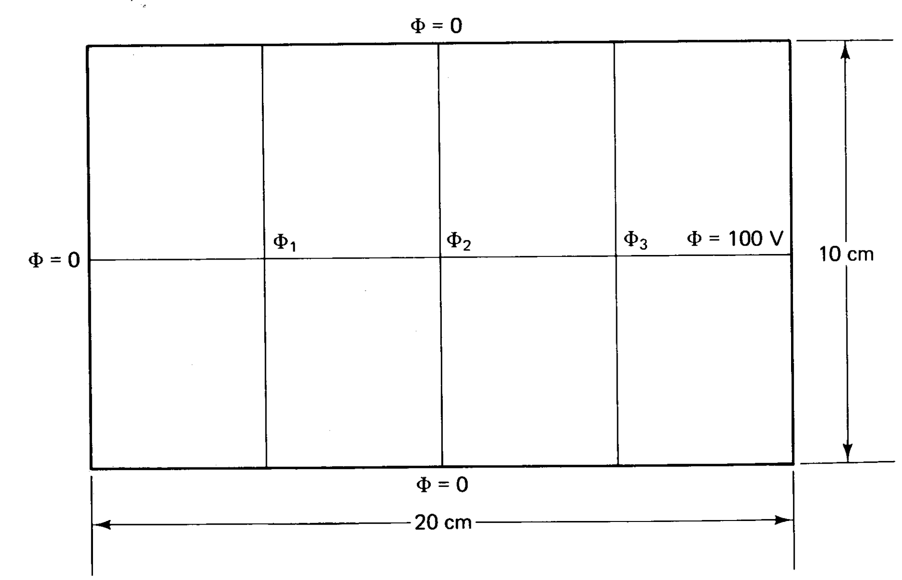
or

$$\Phi_2 - 4\Phi_3 = -100 \tag{4.52}$$



(a)

Figure 4.18a Rectangular geometry and boundary condition for the electric potential problem of example 4.14.



(b)

Figure 4.18b Geometry of 2 × 4 finite difference mesh.

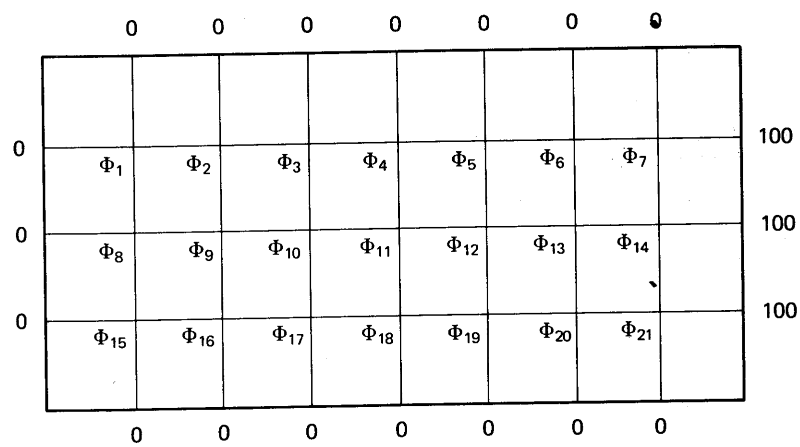


Figure 4.19 Geometry of the $h = 2.5$-cm mesh with twenty-one potential nodes.

4. Equations 4.50 to 4.52 are three equations in the three unknowns, Φ_1, Φ_2, and Φ_3. These three equations may be solved using one of the methods described in calculus courses. The results are

$$\Phi_1 = 1.79, \qquad \Phi_2 = 7.14, \qquad \Phi_3 = 26.79$$

5. With the coarse mesh we used, we do not expect to get accurate final results. Redoing the problem with a smaller value of h should improve the accuracy of the solution. Figure 4.19 shows the mesh geometry for $h = 2.5$ cm, which is half the mesh size used in the previous calculations. In this case, however, we have twenty-one unknown values of the potential at the various nodes.
6. Once again, applying the difference equation of step 2 at the various nodes results in the following 21×21 matrix:

$$\begin{bmatrix}
-4&1&&&&&&1&&&&&&&&&&&&& \\
1&-4&1&&&&&&1&&&&&&&&&&&& \\
&1&-4&1&&&&&&1&&&&&&&&&&& \\
&&1&-4&1&&&&&&1&&&&&&&&&& \\
&&&1&-4&1&&&&&&1&&&&&&&&& \\
&&&&1&-4&1&&&&&&1&&&&&&&& \\
&&&&&1&-4&&&&&&&1&&&&&&& \\
1&&&&&&&-4&1&&&&&&1&&&&&& \\
&1&&&&&&1&-4&1&&&&&&1&&&&& \\
&&1&&&&&&1&-4&1&&&&&&1&&&& \\
&&&1&&&&&&1&-4&1&&&&&&1&&& \\
&&&&1&&&&&&1&-4&1&&&&&&1&& \\
&&&&&1&&&&&&1&-4&1&&&&&&1& \\
&&&&&&1&&&&&&1&-4&&&&&&&1 \\
&&&&&&&1&&&&&&&-4&1&&&&& \\
&&&&&&&&1&&&&&&1&-4&1&&&& \\
&&&&&&&&&1&&&&&&1&-4&1&&& \\
&&&&&&&&&&1&&&&&&1&-4&1&& \\
&&&&&&&&&&&1&&&&&&1&-4&1& \\
&&&&&&&&&&&&1&&&&&&1&-4&1 \\
&&&&&&&&&&&&&1&&&&&&1&-4
\end{bmatrix}
\begin{bmatrix}
\Phi_1 \\ \Phi_2 \\ \Phi_3 \\ \Phi_4 \\ \Phi_5 \\ \Phi_6 \\ \Phi_7 \\ \Phi_8 \\ \Phi_9 \\ \Phi_{10} \\ \Phi_{11} \\ \Phi_{12} \\ \Phi_{13} \\ \Phi_{14} \\ \Phi_{15} \\ \Phi_{16} \\ \Phi_{17} \\ \Phi_{18} \\ \Phi_{19} \\ \Phi_{20} \\ \Phi_{21}
\end{bmatrix}
=
\begin{bmatrix}
0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ -100 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ -100 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ -100
\end{bmatrix}$$

TABLE 4.1 COMPARISON BETWEEN FINITE DIFFERENCE AND ANALYTICAL RESULTS

	Potential values	Percentage error	Potential values	Percentage error	Analytical solution
	(h = 5 cm)		(h = 2.5 cm)		
Φ_9	1.786	63	1.289	17.8	1.094
Φ_{11}	7.143	30	6.019	9.7	5.489
Φ_{13}	26.786	2.7	26.289	0.75	26.094

The solution for the electric potential at the various nodes is given by

$\Phi_1 = 0.353$	$\Phi_8 = 0.499$	$\Phi_{15} = 0.353$
$\Phi_2 = 0.913$	$\Phi_9 = 1.289$	$\Phi_{16} = 0.913$
$\Phi_3 = 2.010$	$\Phi_{10} = 2.832$	$\Phi_{17} = 2.010$
$\Phi_4 = 4.296$	$\Phi_{11} = 6.019$	$\Phi_{18} = 4.296$
$\Phi_5 = 9.153$	$\Phi_{12} = 12.654$	$\Phi_{19} = 9.153$
$\Phi_6 = 19.663$	$\Phi_{13} = 26.289$	$\Phi_{20} = 19.663$
$\Phi_7 = 43.210$	$\Phi_{14} = 53.177$	$\Phi_{21} = 43.210$

Table 4.1 compares the results for h = 5 cm, and h = 2.5 cm. From this comparison, it is clear that the h = 2.5 cm results agree better with the analytical solution available for this simple geometry. As expected, the accuracy of the finite difference results improves with the reduction in the mesh size h. Clearly, any further reduction in h results in a larger-size matrix; hence, a compromise should be made between the desired accuracy and the computational time and effort required.

EXAMPLE 4.15

In the $6 \times 8\ \text{m}^2$ rectangular region shown in Figure 4.20a, the electric potential is zero on the boundaries. The charge distribution, however, is uniform and given by $\rho_v = 2\epsilon_o$. Solve Poisson's equation to determine the potential distribution in the rectangular region.

Solution

To determine the potential distribution in the rectangular region, we use Poisson's equation.

$$\nabla^2 \Phi = -\frac{\rho_v}{\epsilon_o} = -2$$

with zero potential $\Phi = 0$ on the boundaries. By establishing the rectangular grid shown in Figure 4.20b, we realize that we have six nodes and, hence, six unknown potentials for which to solve. Replacing $\nabla^2 \Phi$ by its finite difference representation, we obtain

$$\frac{1}{h^2}(\Phi_{i+1,j} + \Phi_{i-1,j} + \phi_{i,j+1} + \phi_{i,j-1} - 4\phi_{i,j}) + 2 = 0 \quad (4.53)$$

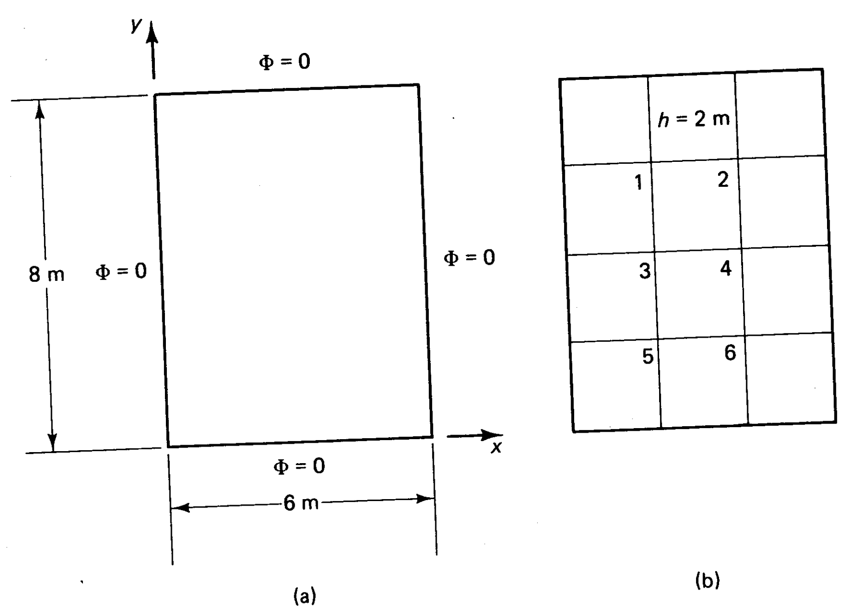


Figure 4.20 Geometry of the $6 \times 8\ m^2$ rectangular region and the $h =$ 2 m mesh.

It should be noted that although the mesh size was not explicitly used in solving Laplace's equation in the previous example, h is included as a part of the matrix formation in solving Poisson's equation. In SI system of units, h should be in meters. By applying the preceding difference equation at the various nodes in Figure 4.20b, we obtain the following matrix equation:

$$\begin{bmatrix} -4 & 1 & 1 & 0 & 0 & 0 \\ 1 & -4 & 0 & 1 & 0 & 0 \\ 1 & 0 & -4 & 1 & 1 & 0 \\ 0 & 1 & 1 & -4 & 0 & 1 \\ 0 & 0 & 1 & 0 & -4 & 1 \\ 0 & 0 & 0 & 1 & 1 & -4 \end{bmatrix} \begin{bmatrix} \Phi_1 \\ \Phi_2 \\ \Phi_3 \\ \Phi_4 \\ \Phi_5 \\ \Phi_6 \end{bmatrix} = \begin{bmatrix} -8 \\ -8 \\ -8 \\ -8 \\ -8 \\ -8 \end{bmatrix}$$

Instead of solving the resulting six equations, we may note some symmetry considerations in Figure 4.20b. It is clear that

$$\Phi_1 = \Phi_2 = \Phi_5 = \Phi_6 \qquad \text{and that} \qquad \Phi_3 = \Phi_4$$

Taking these symmetry considerations into account, the number of equations reduces to two, and we obtain the following solution:

$$\Phi_1 = 4.56, \qquad \Phi_3 = 5.72$$

To improve the accuracy of the potential distribution, finer mesh such as the one shown in Figure 4.21 is required. Because of the large number of nodes in this case, symmetry should be used, and a solution for only one-quarter of the rectangular geometry is desired. The application of the difference equation at nodes 1, 2, 4, and 5 should proceed

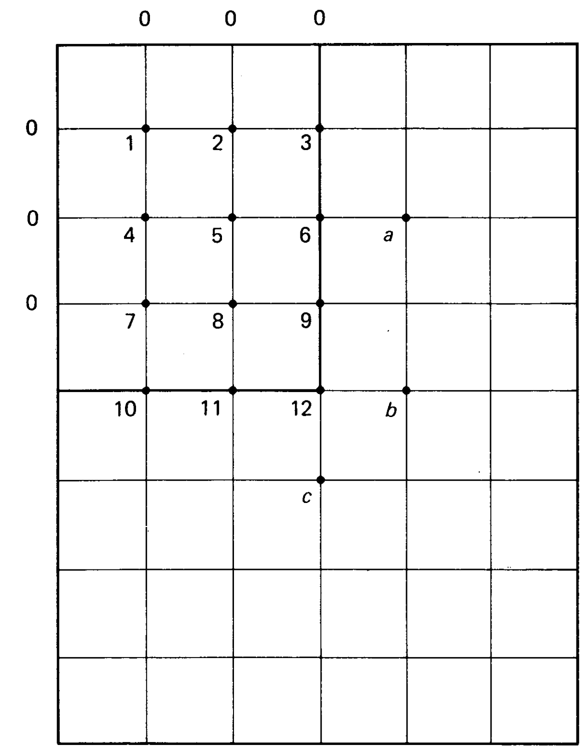


Figure 4.21 The finer mesh solution and symmetry consideration of example 4.15.

routinely, whereas special care should be exercised at the boundary nodes 3, 6, 9, 10, and 11, and also at the corner node 12.

For example, applying the difference equation at node 6 yields

$$\frac{1}{h^2}(\Phi_a + \Phi_3 + \Phi_9 + \Phi_5 - 4\Phi_6) + 2 = 0$$

or

$$\frac{1}{h^2}(\Phi_3 + \Phi_9 + 2\Phi_5 - 4\Phi_6) + 2 = 0 \tag{4.54}$$

In equation 4.54, symmetry was used to complete the five-point star difference equation. Specifically the potential at node a to the right of 6 was taken equal to Φ_5. Similarly at the corner node 12, we obtain

$$\frac{1}{h^2}(\Phi_{11} + \Phi_9 + \Phi_b + \Phi_c - 4\Phi_{12}) + 2 = 0$$

Because of symmetry, $\Phi_{11} = \Phi_b$ and $\Phi_9 = \Phi_c$, hence,

$$\frac{1}{h^2}(2\Phi_{11} + 2\Phi_9 - 4\Phi_{12}) + 2 = 0$$

The matrix equation for the twelve nodes shown in Figure 4.21 is then

$$\begin{bmatrix} -4 & 1 & & 1 & & & & & & & & \\ 1 & -4 & 1 & & 1 & & & & & & & \\ & 2 & -4 & & & 1 & & & & & & \\ 1 & & & -4 & 1 & & 1 & & & & & \\ & 1 & & 1 & -4 & 1 & & 1 & & & & \\ & & 1 & & 2 & -4 & & & 1 & & & \\ & & & 1 & & & -4 & 1 & & 1 & & \\ & & & & 1 & & 1 & -4 & 1 & & 1 & \\ & & & & & 1 & & 2 & -4 & & & 1 \\ & & & & & & 2 & & & -4 & 1 & \\ & & & & & & & 2 & & 1 & -4 & 1 \\ & & & & & & & & 2 & & 2 & -4 \end{bmatrix} \begin{bmatrix} \Phi_1 \\ \Phi_2 \\ \Phi_3 \\ \Phi_4 \\ \Phi_5 \\ \Phi_6 \\ \Phi_7 \\ \Phi_8 \\ \Phi_9 \\ \Phi_{10} \\ \Phi_{11} \\ \Phi_{12} \end{bmatrix} = \begin{bmatrix} -2 \\ -2 \\ -2 \\ -2 \\ -2 \\ -2 \\ -2 \\ -2 \\ -2 \\ -2 \\ -2 \\ -2 \end{bmatrix} \tag{4.55}$$

The "2" coefficient in the coefficient matrix (to the left) of equation (4.55) appears whenever symmetry consideration is used at boundary and corner nodes. It should be noted that the 12×12 coefficient matrix in equation 4.55 is the same for both Laplace's and Poisson's equations. The constant vector on the right-hand side of equation 4.55, however, depends on the charge distribution within and the potential at the boundaries of the region of interest. Furthermore, if instead of a uniform charge distribution we have a given charge distribution $\rho_v(x, y)$, the constants vector on the right-hand side of equation 4.55 should reflect the value of $\rho_v(x, y)$ calculated at each node. Solution of equation 4.55 gives.

$$\Phi_1 = 2.04, \quad \Phi_2 = 3.05, \quad \Phi_3 = 3.35$$

$$\Phi_4 = 3.12, \quad \Phi_5 = 4.79, \quad \Phi_6 = 5.32$$

$$\Phi_7 = 3.66, \quad \Phi_8 = 5.69, \quad \Phi_9 = 6.34$$

$$\Phi_{10} = 3.82, \quad \Phi_{11} = 5.96, \quad \Phi_{12} = 6.65$$

4.8.3 Capacitance Calculation Using Finite Difference

The question that is often asked by students is how to relate the calculated values of the node potential to quantities of engineering interest, such as the capacitance of a system of conductors of complex geometry. Calculation of the capacitance from the obtained potential distribution may be done through Gauss's law, as follows:

$$\oint_s \epsilon \mathbf{E} \cdot d\mathbf{s} = q = -\oint_s \epsilon \nabla\Phi \cdot d\mathbf{s} \tag{4.56}$$

In equation 4.56, the electric field $\mathbf{E}$ was substituted by the electric potential $\mathbf{E} = -\nabla\Phi$. In a two-dimensional problem such as the conductor arrangement shown in Figure 4.22, the potential is independent of the axial coordinate, and the closed surface s may be replaced by the closed contour c. Hence,

$$-\oint_s \epsilon \nabla\Phi \cdot d\mathbf{c} = -\oint_c \epsilon \frac{\partial \Phi}{\partial n} dc = q \tag{4.57}$$

where q in this case is the charge per unit length in coulombs per meter. $\nabla\Phi \cdot d\mathbf{c}$ in equation 4.57 is replaced by the normal derivative $\partial\Phi/\partial n$ integrated over the closed